Continuity (cont.)

Recall: If (X, d_X) and (Y, d_Y) are metric spaces, $E \subseteq X$, $p \in E$, and $f: E \to Y$, then we say f is continuous at p if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in E$ with $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \varepsilon$; and we write $\lim_{x \to p} f(x) = f(p)$

Also, if f is continuous at p for every $p \in E$, then we say f is continuous on E.

Let's look at an example of how to use this definition to prove something.

Thm I: Suppose X, Y, Z are metric spaces,

$$f: X \to Y, \quad g: f(X) \to Z.$$

If f is continuous at $p \in X$ and g is continuous at $f(p) \in f(X)$, then $g \circ f$ is continuous at p.

Proof: We need to show $g \circ f$ is continuous at p. So let $\varepsilon > 0$. Since g is continuous at $f(p)$, there exists $\eta > 0$ such that

if $y \in f(X)$ with $d_Y(y, f(p)) < \eta$

then $d_Z(g(y), g(f(p))) < \varepsilon$.

Furthermore, since f is continuous at p, there exists $\delta > 0$ such that

if $x \in X$ with $d_X(x, p) < \delta$

then $d_Y(f(x), f(p)) < \eta$

Thus, we have that, for $x \in X$:

$$\begin{aligned} d_X(x, p) < \delta &\Rightarrow d_Y(f(x), f(p)) < \eta \\ &\Rightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon \\ &\Rightarrow d_Z((g \circ f)(x), (g \circ f)(p)) < \varepsilon. \end{aligned}$$

Therefore, $g \circ f$ is continuous at p. ∎

One consequence of this theorem:

Yesterday, you proved on your worksheet that $\sin x$ is continuous on all of $\mathbb{R}$.

In addition, it is not difficult to show that any linear function is continuous on $\mathbb{R}$.

Therefore, by Thm I, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ is continuous on $\mathbb{R}$ since it is the composition of $\sin x$ with $x + \frac{\pi}{2}$.

Properties of Limits

Suppose $E \subseteq X$ a metric space, p a limit point of E, $f, g : E \to \mathbb{R}$, and $\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$.

Then (a) $\lim_{x \to p} (f+g)(x) = A + B$

(b) $\lim_{x \to p} (f-g)(x) = A - B$

(c) $\lim_{x \to p} (fg)(x) = AB$

(d) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$.

Consequences of these properties:

If f and g are continuous on E, then so is their sum, difference, product, and quotient (in this last case, only at points $x \in E$ where $g(x) \neq 0$).

Ex: $\tan x = \dfrac{\sin x}{\cos x}$ is continuous wherever $\cos x \neq 0$, i.e. when $x \neq \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$