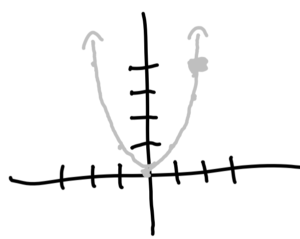
Continuity

For the past several days, we have talked about limits/convergence of sequences, i.e. limits of functions $f: \mathbb{N} \to \mathbb{R}$ as you let $n \to \infty$.

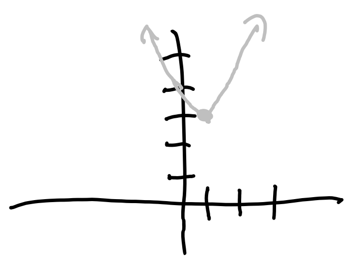
What if we switched instead to functions $\mathbb{R} \to \mathbb{R}$ and talked about limits as $x \to p$, or more generally, limits of functions between two metric spaces?

Ex: 1 $f(x) = x^2$

$$\lim_{x \to 2} x^2 = 4$$

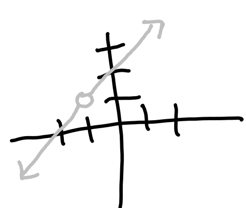


2 $f(x) = \begin{cases} 2x+1 & x \geq 1 \\ x^2 - 2x + 4 & x < 1 \end{cases}$



$$\lim_{x \to 1} f(x) = 3$$

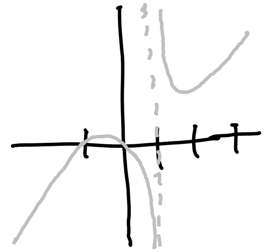
3 $f(x) = \dfrac{x^2 + 3x + 2}{x+1}$, $x \neq -1$



$$\lim_{x \to -1} f(x) = 1$$

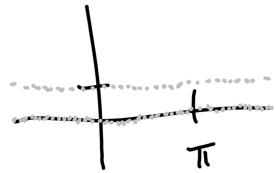
(4) $f(x) = \dfrac{x^2+x}{x-1}, \; x \neq 1$

$\lim\limits_{x \to 1} f(x)$ does not exist



(5) $f(x) = \begin{cases} 0 & \text{if } x \text{ rational} \\ 1 & \text{if } x \text{ irrational} \end{cases}$

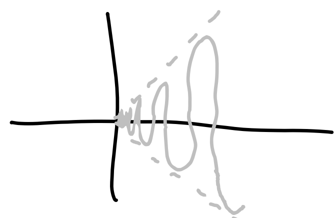
$\lim\limits_{x \to \pi} f(x)$ does not exist



(6) $f : (0, \infty) \to \mathbb{R}$

$x \mapsto x \sin\left(\frac{1}{x}\right)$

$\lim\limits_{x \to 0} f(x) = 0$



Let's look at the actual definition of convergence/limits for a function between two metric spaces.

But before we do that, we need to define a new concept:

<u>Def 1:</u> Let X be a metric space, $p \in X$, $E \subseteq X$. Then p is a <u>limit point</u> of E if for every $r > 0$, the ball $B(p, r)$ contains a point

$q \neq p$ with $q \in E$. In other words,

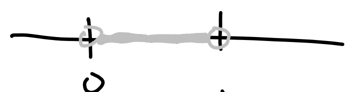
$$(B(p,r) \setminus \{p\}) \cap E \neq \emptyset.$$

Or equivalently, there is $q \in E$ with $q \neq p$ and $d(p,q) < r$.

Examples: (1) 0 is a limit point of the interval $(0,1)$, since the ball $B(0,r)$ contains $\frac{r}{2}$ which is in $(0,1)$.

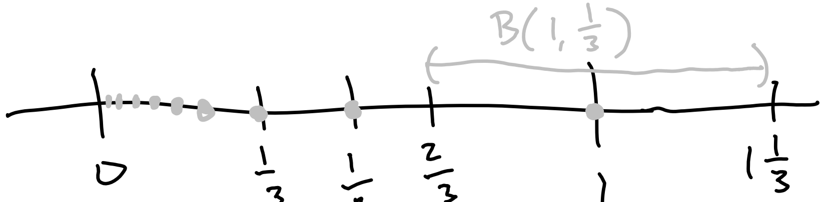
Similarly, so is 1.

Note so is every $p \in (0,1)$.



(2) 0 is also a limit point of the set $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, since for any $r > 0$, there exists $N \in \mathbb{N}$ with $N > \frac{1}{r} \Rightarrow \frac{1}{N} < r$ and so $\frac{1}{N} \in B(0,r)$ with $\frac{1}{N} \neq 0$ and $\frac{1}{N} \in E$.

However, 1 is not a limit point of E, since $B(1, \frac{1}{3}) \cap E = \{1\}$.



Idea: p is a limit point of E if there are points in E that are arbitrarily close to p.

Now we are ready to define the limit of a function.

Def 2: Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subseteq X$, p a limit point of E, $f: E \to Y$, and $q \in Y$. We save <u>f converges to q at p</u> and write $f(x) \to q$ as $x \to p$, or $\lim_{x \to p} f(x) = q$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in E$ with $0 < d_X(x, p) < \delta$, then $d_Y(f(x), q) < \varepsilon$.

"Near p, the function stays close to q."

We require p to be a limit point in the definition of convergence so that we can talk about behavior of f arbitrarily close to p.

Examples: 1 Prove $\lim_{x \to 2} x^2 = 4$.

Proof: Let $\varepsilon > 0$. We wish to find $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|x^2 - 4| < \varepsilon$.

So if we have $0 < |x-2| < \delta$, then

$$|x^2 - 4| = |x+2|\,|x-2| = |x-2+4|\,|x-2|$$
$$\leq (|x-2| + 4)\,|x-2| < (\delta + 4)\delta$$
$$< 5\delta \text{ if } \delta < 1$$
$$< \varepsilon \text{ if } \delta < \frac{\varepsilon}{5} \quad \text{since } 5\delta < \varepsilon \iff \delta < \frac{\varepsilon}{5}$$

Thus, if $\delta < 1$ and $\frac{\varepsilon}{5}$, say $\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{10}\right)$, then $0 < |x-2| < \delta \implies |x^2 - 4| < \varepsilon$. So $\lim_{x \to 2} x^2 = 4$.

[illegible]

(2) Prove that for $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=x\sin\left(\frac{1}{x}\right)$, $\lim_{x\to 0} f(x)=0$.

<u>Proof:</u> Let $\varepsilon>0$. If $x\in(0,\infty)$ with $0<|x-0|=|x|=x<\delta$ for $\delta>0$, then we have

$$|x\sin(\tfrac{1}{x})-0| = |x\sin(\tfrac{1}{x})| = x\,|\sin(\tfrac{1}{x})|$$
$$\leq x < \delta$$

Thus, if we take $\delta<\varepsilon$, say $\delta=\frac{\varepsilon}{2}$, then if $x\in(0,\infty)$ with $x<\delta$, we have $|x\sin(\frac{1}{x})|<\varepsilon$.

Note that sometimes a function achieves its limit, e.g. $\lim_{x\to 2} x^2 = 4 = 2^2$. This phenomenon is called continuity.

<u>Def 3:</u> Let (X,d_X) and (Y,d_Y) be metric spaces, $E\subseteq X$, $p\in E$, and $f:E\to Y$. Then f is <u>continuous at p</u> if for every $\varepsilon>0$ there exists a $\delta>0$ such that if $x\in E$ with $d_X(x,p)<\delta$, then $d_Y(f(x),f(p))<\varepsilon$. In other words,

$$\lim_{x \to p} f(x) = f(p).$$

If f is continuous at every point in E, then we say that f is <u>continuous on E</u>.

- <u>Note</u>: When we talk about continuity at a point p, we require p to be in the domain E of the function, rather than just a limit point of E.

<u>Examples</u>: ① As we just showed, $\lim_{x \to 2} x^2 = 4 = 2^2$ so $f(x) = x^2$ is continuous at 2. In fact, one can show it is continuous at every $x \in \mathbb{R}$.

② If we altered our second example to be the function $f : [0, \infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \end{cases}$$

then we showed this function is continuous at 0.

(3) The function $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ given by (8)

$f(x) = \frac{1}{x}$ is continuous at every point on its domain, even though $\lim_{x \to 0} f(x)$ does not exist.