1. Prove using the definition that the following sequence is Cauchy:
$a_1 = 1$, $a_2 = 2$, and $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geq 3$.

<u>Proof:</u> We first show by induction that $|a_n - a_{n-1}| = \frac{1}{2^{n-2}}$ for $n \geq 2$.

First, we have that $|a_2 - a_1| = |2-1| = 1 = \frac{1}{2^{2-2}}$.

So the base case holds.

Next, suppose that $|a_n - a_{n-1}| = \frac{1}{2^{n-2}}$ for some $n \geq 2$.

Then we have

$$
\begin{aligned}
|a_{n+1} - a_n| &= \left|\tfrac{1}{2}(a_n + a_{n-1}) - a_n\right| \quad \text{since } n+1 \geq 3 \\
&= \left|-\tfrac{1}{2}a_n + \tfrac{1}{2}a_{n-1}\right| \\
&= \tfrac{1}{2}|a_n - a_{n-1}| \\
&= \tfrac{1}{2}\left(\tfrac{1}{2^{n-2}}\right) \quad \text{by our inductive hypsthesis} \\
&= \frac{1}{2^{n-1}} = \frac{1}{2^{n+1-2}}
\end{aligned}
$$

Therefore, $|a_n - a_{n-1}| = \frac{1}{2^{n-2}}$ for every $n \geq 2$.

Now we show the sequence is Cauchy.

Let $\varepsilon > 0$. Then for $m, n \in \mathbb{N}$, we have

$$d(a_m, a_n) = |a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - \cdots - a_{n+1} + a_{n+1} - a_n|$$

$$\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n|$$

$$= \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}} \left(\frac{1}{2^{m-n-1}} + \frac{1}{2^{m-n-2}} + \cdots + \frac{1}{2} + 1 \right)$$

$$< \frac{1}{2^{n-1}} (2) = \frac{1}{2^{n-2}}.$$

Thus we need to pick $N \in \mathbb{N}$ such that

$$\frac{1}{2^{N-2}} < \varepsilon \iff \frac{1}{\varepsilon} < 2^{N-2} \iff -\log_2 \varepsilon < N-2$$

$$\iff N > 2 - \log_2 \varepsilon.$$

Then if $m, n \geq N$, $\frac{1}{2^{n-2}} \leq \frac{1}{2^{N-2}}$, and therefore

$$d(a_m, a_n) < \frac{1}{2^{n-2}} \leq \frac{1}{2^{N-2}} < \varepsilon.$$

So the sequence is Cauchy.

$\boxed{\text{\&}}$

Are the following metric spaces complete?

2. The interval $(0,1)$ under the usual metric $d(a,b) = |b-a|$

No. We have seen the sequence $a_n = \frac{1}{n}$ is Cauchy, and this sequence lives in $(0,1)$. However, $\frac{1}{n} \to 0$ but $0 \notin (0,1)$.

3. The integers $\mathbb{Z}$ under the usual metric $d(a,b) = |b-a|$.

Yes. Let $\{a_n\} \subseteq \mathbb{Z}$ be Cauchy. Then for $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that when $m,n \geq N$, then $d(a_m, a_n) = |a_m - a_n| < \frac{1}{2}$. But $a_m, a_n \in \mathbb{Z}$, so $|a_m - a_n| \in \mathbb{Z} \Rightarrow |a_m - a_n| = 0 \Rightarrow a_m = a_n$.

Thus, the sequence $\{a_n\}$ eventually becomes constant, so $\{a_n\}$ converges to that constant. So every Cauchy sequence in $\mathbb{Z}$ converges in $\mathbb{Z}$, so $\mathbb{Z}$ is complete. $\boxtimes$

Let $\{a_n\}$ be a Cauchy sequence in a metric space (X, d). Prove the following.

4. $\{a_n\}$ is bounded.

Let $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $d(a_m, a_n) < 1$ for every $m, n > N$.

Then let $M = \max\{d(a_1, a_N), d(a_2, a_N), \dots, d(a_{N-1}, a_N), 1\}$.

Then we have that

$$d(a_n, a_N) \leq M < M+1 \text{ if } n < N$$
$$d(a_n, a_N) < 1 < M+1 \text{ if } n \geq N.$$

Therefore $\{a_n\} \subseteq B(a_N, M+1)$. So $\{a_n\}$ is bounded. $\blacksquare$

5. If $\{a_n\}$ has a convergent subsequence, then $\{a_n\}$ converges.

<u>Proof:</u> Let $\{a_n\}$ be Cauchy and $\{a_{n_k}\}$ a subsequence such that $a_{n_k} \to a$ for some $a \in X$. We wish to show $a_n \to a$.

Let $\varepsilon > 0$. Since $\{a_n\}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that $d(a_n, a_m) < \varepsilon$ for $m, n > N_1$. And since $a_{n_k} \to a$, there exists $N_2 \in \mathbb{N}$ such that $d(a_{n_k}, a) < \varepsilon$ when $n_k > N_2$.

Then we have:

$$
\begin{aligned}
d(a_n, a) &\leq d(a_n, a_{n_k}) + d(a_{n_k}, a) \\
&< \varepsilon + d(a_{n_k}, a) \quad \text{when } n, n_k > N_1 \\
&< \varepsilon + \varepsilon \quad \text{when } n_k > N_2 \\
&= 2\varepsilon.
\end{aligned}
$$

Therefore $a_n \to a$. $\boxed{m}$

6. Use Problems 4 and 5 and the Bolzano-Weierstrass Theorem to prove $\mathbb{R}$ is complete.

Proof: Let $\{a_n\} \subseteq \mathbb{R}$ be Cauchy. By Problem 7, $\{a_n\}$ is bounded. By the Bolzano-Weierstrass Theorem, $\{a_n\}$ has a convergent subsequence. By Problem 8, this implies $\{a_n\}$ converges.

Therefore, $\mathbb{R}$ is complete. $\blacksquare$