Cauchy Sequences

Yesterday, we defined what it means for a sequence to be Cauchy.

Idea: A sequence is Cauchy if eventually all terms in the sequence are arbitrarily close to each other.

Def 1: A sequence $\{a_n\}$ in a metric space (X,d) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m,n \geq N \Rightarrow d(a_m, a_n) < \varepsilon$.

Examples: 1 Yesterday, we showed the sequences $\{\frac{1}{n}\}$ and $\{\frac{n+1}{n}\}$ were Cauchy. You also showed on your worksheet that $\{\frac{4}{n^5}\}$ and $\left\{\frac{(-1)^n + n^2}{n^2+1}\right\}$ were Cauchy.

A nonexample:

(2) Just because terms in a sequence get arbitrarily close to the next term in the sequence does not mean that the sequence is Cauchy.

For example, let $a_n = \sqrt{n}$. Then for $\varepsilon > 0$,

$$d(\sqrt{n+1}, \sqrt{n}) = |\sqrt{n+1} - \sqrt{n}| = \left| \frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right| = \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} \right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}}$$

$$= \frac{1}{2\sqrt{n}} < \varepsilon \text{ when } n > \frac{1}{4\varepsilon^2}$$

scratch work: $\frac{1}{2\sqrt{n}} < \varepsilon \iff 1 < 2\varepsilon\sqrt{n}$

$\iff \sqrt{n} > \frac{1}{2\varepsilon}$

$\iff n > \frac{1}{4\varepsilon^2}$

Thus consecutive terms get arbitrarily close to each other.

However, the sequence $\sqrt{n}$ gets arbitrarily large, so this is going to make the sequence not Cauchy:

Not Cauchy means there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there exists $m, n \geq N$ such that $d(a_m, a_n) > \varepsilon$

Consider $\varepsilon = 1$, and let $N \in \mathbb{N}$. Then note, if we let $M > (1+\sqrt{N})^2$

$$d(\sqrt{M}, \sqrt{N}) = |\sqrt{M} - \sqrt{N}| = \sqrt{M} - \sqrt{N} > 1 + \sqrt{N} - \sqrt{N}$$
$$= 1$$

scratch work: $\sqrt{M} - \sqrt{N} > 1 \iff \sqrt{M} > 1 + \sqrt{N}$
$\iff M > (1+\sqrt{N})^2$

So $\sqrt{n}$ is not Cauchy.

Notice that all of our examples of Cauchy sequences were convergent sequences ($\frac{1}{n} \to 0$, $\frac{n+1}{n} \to 1$, $\frac{4}{n^5} \to 0$, $\frac{(-1)^n + n^2}{n^2+1} \to 1$), while our example of a sequence which was not Cauchy did not converge ($\sqrt{n} \to \infty$)

Are convergent sequences always Cauchy? And do Cauchy sequences always converge?

Thm I: Every convergent sequence in a metric space (X, d) is Cauchy.

Proof: Let $\{a_n\}$ be a sequence in X such that $a_n \to a$. We wish to show $\{a_n\}$ is Cauchy. So let $\varepsilon > 0$. Since $a_n \to a$, we know there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(a_n, a) < \frac{\varepsilon}{2}$.

Then let $m, n \geq N$. We have, by the triangle inequality,

$$d(a_m, a_n) \leq d(a_m, a) + d(a, a_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\{a_n\}$ is Cauchy. $\blacksquare$

So a_n convergent $\Rightarrow$ a_n Cauchy.

Does a_n Cauchy $\Rightarrow$ a_n convergent?

Not necessarily.

Note that $\mathbb{Q}$ is a metric space with the metric $d(a,b) = |b-a|$

Consider the sequence in $\mathbb{Q}$ given by

$1.4,\ 1.41,\ 1.414,\ \ldots$

i.e., a_n is $\sqrt{2}$ up to the first n decimal places, or more precisely, $a_n = \dfrac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$

where $\lfloor x \rfloor$ is the largest integer n with $n \leq x$.

This sequence is Cauchy:

Let $\varepsilon > 0$. Then pick $N \in \mathbb{N}$ with $10^{-N} < \varepsilon$

Then let $m, n \geq N$. Notice then that the first N decimal places of a_m and a_n are the same, since they both contain at least the first N decimals of $\sqrt{2}$.

$10^{-N} < \varepsilon \iff \varepsilon^{-1} < 10^{N} \iff -\log_{10}\varepsilon < N$

Then the first N decimal places of $a_m - a_n$ are 0. Thus $|a_m - a_n| < 10^{-N}$

Therefore,

$$d(a_m, a_n) = |a_m - a_n| < 10^{-N} < \varepsilon$$

$\{ 0.01 < 0.1 = 10^{-1}$
$0.0042 < 0.01 = 10^{-2}$
etc.

Thus, this sequence is Cauchy.

However, notice how as a sequence in $\mathbb{R}$, this sequence converges to $\sqrt{2}$ which is not rational. So this sequence does not converge in $\mathbb{Q}$.

Def 2: A metric space (X, d) where every Cauchy sequence converges to a point in X is called complete.

Thus, we have shown that $\mathbb{Q}$ is not complete.

However, $\boxed{\text{Thm II: } \mathbb{R} \text{ is complete.}}$

Proof: On Worksheet