More on Sequences

Thm I: If $\{a_n\}$ and $\{b_n\}$ are bounded sequences in $\mathbb{R}$, then $\{a_n+b_n\}$ is bounded.

Proof: Suppose $\{a_n\}$ and $\{b_n\}$ are bounded.

Then $\{a_n\} \subseteq B(p,r)$ and $\{b_n\} \subseteq B(q,s)$

for some $p,q \in \mathbb{R}$, $r,s>0$.

We wish to show $\{a_n+b_n\}$ is bounded,

and claim that $\{a_n+b_n\} \subseteq B(p+q, r+s)$

Indeed, we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} d(a_n+b_n, p+q) &= |a_n+b_n-p-q| \\ &= |a_n-p+b_n-q| \\ &\leq |a_n-p|+|b_n-q| \quad (\triangle \text{ inequality}) \\ &= d(a_n,p)+d(b_n,q) \\ &< r+s \quad \text{since } a_n \in B(p,r), \; b_n \in B(q,s) \end{aligned}$$

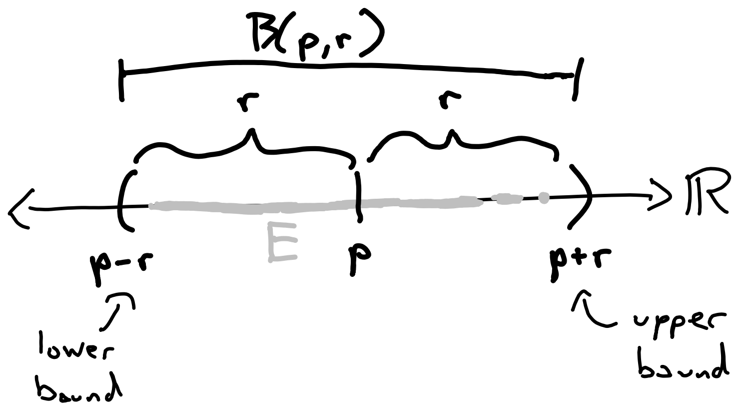
Thus $\{a_n+b_n\} \subseteq B(p+q, r+s)$, and $\{a_n+b_n\}$ is bounded. ∎

Thm II: Every sequence with at least one convergent subsequence has infinitely many convergent subsequences.

Proof: Let $\{a_n\}$ be a sequence and $\{a_{n_k}\}$ a subsequence such that $a_{n_k} \to a$. Then we know any subsequence of a_{n_k} converges to a (a subsequence is itself a sequence!), and any subsequence of $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. Since $\{a_{n_k}\}$ has infinitely many subsequences, this implies there are infinitely many subsequences of a_n which converge to a. $\blacksquare$

Thm III: If $\{a_n\}$ is a bounded and monotonic real sequence, then $\{a_n\}$ converges.

Lemma: Let $E \subseteq \mathbb{R}$. Then E is bounded (i.e., $E \subseteq B(p,r)$ for some $p, r \in \mathbb{R}$ with $r > 0$) if and only if E is bounded (i.e. there exist $a, b \in \mathbb{R}$ such that for every $x \in E$, $a \leq x \leq b$).



Pf of Thm III: We will prove the case when $\{a_n\}$ is increasing. So let $\{a_n\}$ be increasing and bounded. By our Lemma, a_n is bounded above, so by the least upper bound property of $\mathbb{R}$, $\{a_n\}$ has a supremum $a \in \mathbb{R}$. So $\forall n \in \mathbb{N}$,

$$a_n \leq a$$

In addition, for any $\varepsilon > 0$, $a - \varepsilon < a$, so $\exists\, N \in \mathbb{N}$ with $a - \varepsilon < a_N$.

Then for any $n > N$, we have, since $\{a_n\}$ is increasing,

$$a - \varepsilon < a_N \leq a_n \leq a < a + \varepsilon$$

$$\Rightarrow a - \varepsilon < a_n < a + \varepsilon$$

$$\Rightarrow -\varepsilon < a_n - a < \varepsilon$$

$$\Rightarrow |a_n - a| < \varepsilon$$

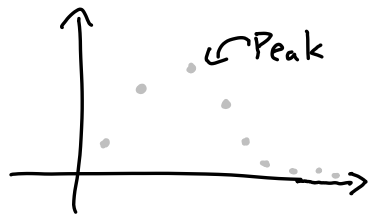
$$\Rightarrow d(a_n, a) < \varepsilon$$

Thus, given $\varepsilon > 0$, we have found $N \in \mathbb{N}$ such that $d(a_n, a) < \varepsilon$ when $n > N$.

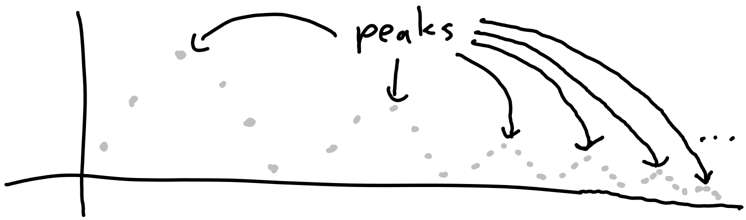
So $a_n \to a$. $\blacksquare$

Thm IV: Every sequence in $\mathbb{R}$ has a monotone subsequence

Proof: Let $\{a_n\}$ be a sequence in $\mathbb{R}$.
We will call $n \in \mathbb{N}$ a peak of $\{a_n\}$ if $n < m$ implies $a_n > a_m$.



How many peaks does $\{a_n\}$ have?



Maybe infinite!

Suppose $\{a_n\}$ has infinitely many peaks $n_1 < n_2 < n_3 < \cdots$. Then the subsequence $\{a_{n_k}\}$ corresponding to these peaks is decreasing since $n_k < n_{k+1} \Rightarrow a_{n_k} > a_{n_{k+1}}$.

So now suppose there are only finitely many peaks, and let N be the last peak.

Set $n_1 = N+1$. Then $n_1 > N$, so n_1 is not a peak. So there exists $n_1 < n_2$ with $a_{n_1} \leq a_{n_2}$. Again, $n_2 > N$, so n_2 is not a peak. So there exists $n_2 < n_3$ with $a_{n_2} \leq a_{n_3}$.

Repeating, on the k^{th} step, we will have $n_k > N$ not a peak, so there exists $n_k < n_{k+1}$ with $a_{n_k} \leq a_{n_{k+1}}$.

Thus we have constructed a subsequence $\{a_{n_k}\}$ which is increasing.

Therefore, in either case, $\{a_n\}$ has a monotonic subsequence.

$\boxed{\eta}$

$\boxed{\begin{array}{l}\underline{\text{Thm V}}\text{: (Bolzano-Weierstrass Theorem)} \\ \text{Every bounded sequence has a convergent subsequence.}\end{array}}$

<u>Proof</u>: Let $\{a_n\}$ be bounded. By Theorem $\overline{\text{IV}}$, $\{a_n\}$ has a monotone subsequence $\{a_{n_k}\}$. But since $\{a_n\}$ is bounded, so is $\{a_{n_k}\}$. Thus by Theorem $\overline{\text{III}}$, $\{a_{n_k}\}$ converges. $\blacksquare$