Cauchy Sequences 1 Solutions

Prove using the definition that the following sequences are Cauchy.

1. $a_n = \frac{4}{n^5}$

<u>Proof:</u> Let $\varepsilon > 0$. Then we have

$$d\left(\frac{4}{m^5}, \frac{4}{n^5}\right) = \left|\frac{4}{m^5} - \frac{4}{n^5}\right| \le \frac{4}{m^5} + \frac{4}{n^5} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

when $m, n > \sqrt[5]{\frac{8}{\varepsilon}}$

since $\frac{4}{m^5} < \frac{\varepsilon}{2} \iff \frac{2 \cdot 4}{\varepsilon} < m^5 \iff m > \sqrt[5]{\frac{8}{\varepsilon}}$,

similarly for n.

Thus, $a_n = \frac{4}{n^5}$ is Cauchy. $\boxed{\text{m}}$

2. $a_n = \frac{(-1)^n + n^2}{n^2+1}$

Proof: Let $\varepsilon > 0$. Then we have

$$d\left(\frac{(-1)^m + m^2}{m^2+1}, \frac{(-1)^n + n^2}{n^2+1}\right) = \left|\frac{(-1)^m + m^2}{m^2+1} - \frac{(-1)^n + n^2}{n^2+1}\right|$$

$$= \left|\frac{(-1)^m + m^2}{m^2+1} - 1 + 1 - \frac{(-1)^n + n^2}{n^2+1}\right|$$

$$\leq \left|\frac{(-1)^m + m^2}{m^2+1} - 1\right| + \left|1 - \frac{(-1)^n + n^2}{n^2+1}\right|$$

$$= \left|\frac{(-1)^m + m^2 - m^2 - 1}{m^2+1}\right| + \left|\frac{n^2 + 1 - (-1)^n - n^2}{n^2+1}\right|$$

$$= \left|\frac{(-1)^m - 1}{m^2+1}\right| + \left|\frac{1 - (-1)^n}{n^2+1}\right|$$

Note $|(-1)^m - 1| = 0$ or 2 depending on if m is even or odd

$$\leq \frac{2}{m^2+1} + \frac{2}{n^2+1} \leq \frac{2}{m^2} + \frac{2}{n^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

when $m, n > \frac{2}{\sqrt{\varepsilon}}$ since

$$\frac{2}{m^2} < \frac{\varepsilon}{2} \iff 4 < \varepsilon m^2 \iff m^2 > \frac{4}{\varepsilon} \iff m > \frac{2}{\sqrt{\varepsilon}}$$

and similarly for n.

Thus, $a_n = \frac{(-1)^n + n^2}{n^2+1}$ is Cauchy. $\blacksquare$