Cauchy Sequences

Last week, we learned about sequences $\{a_n\}$ which are functions $f: \mathbb{N} \to X$ metric space

$$n \mapsto a_n$$

e.g. $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

<u>Recall</u>: A sequence $\{a_n\}$ in a metric space (X, d) <u>converges</u> to $a \in X$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(a_n, a) < \varepsilon$

"the sequence gets arbitrarily close to a"

What if instead of the sequence getting arbitrarily close to a point in X, the points in the sequence get arbitrarily close to each other?

<u>Def 1</u>: A sequence $\{a_n\}$ in a metric space (X, d) is called a <u>Cauchy sequence</u> if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \Longrightarrow d(a_m, a_n) < \varepsilon$.

<u>Examples</u>: ① The sequence $\{\frac{1}{n}\}$:

Let $\varepsilon > 0$. Then $d(\frac{1}{m}, \frac{1}{n}) = |\frac{1}{m} - \frac{1}{n}|$

$$\leq |\tfrac{1}{m}| + |-\tfrac{1}{n}| = \frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ when } m, n > \frac{2}{\varepsilon}.$$

side work: $\frac{1}{n} < \frac{\varepsilon}{2} \Longleftrightarrow 1 < \frac{\varepsilon}{2} n \Longleftrightarrow n > \frac{2}{\varepsilon}$

and $\frac{1}{m} < \frac{\varepsilon}{2} \Longleftrightarrow 1 < \frac{\varepsilon}{2} m \Longleftrightarrow m > 2/\varepsilon$

Therefore, $\frac{1}{n}$ is Cauchy.

② The sequence $\{\frac{n+1}{n}\}$: ← Note $\frac{n+1}{n} \to 1$ We can use this to help us!

Let $\varepsilon > 0$. Then

$$d\left(\frac{m+1}{m}, \frac{n+1}{n}\right) = \left|\frac{m+1}{m} - \frac{n+1}{n}\right|$$

$$= \left|\frac{m+1}{m} - 1 + 1 - \frac{n+1}{n}\right|$$

$$\leq \left|\frac{m+1}{m} - 1\right| + \left|1 - \frac{n+1}{n}\right|$$

$$= \left|\frac{m+1-m}{m}\right| + \left|\frac{n-n-1}{n}\right|$$

$$= \left|\frac{1}{m}\right| + \left|\frac{-1}{n}\right|$$

$$= \frac{1}{m} + \frac{1}{n} < \varepsilon \text{ when } m, n > \frac{2}{\varepsilon}$$

as we showed before

Thus, the sequence $\frac{n+1}{n}$ is Cauchy.