# Ordered Sets and Bounds 1 Solutions

Determine whether or not the following relations are orders.

1. $(E, \ll)$ where $(S, <)$ is an ordered set, $E$ is a subset of $S$, and for $x, y \in E$, $x \ll y$ if $x < y$.

Yes! First, if $x, y \in E$, then $x, y \in S$, and so exactly one of $x < y$, $x = y$, or $y < x$ is true. So then exactly one of $x \ll y$, $x = y$, or $y \ll x$ is true.

Second, if $x, y, z \in E$ with $x \ll y$ and $y \ll z$, then this means $x < y$ and $y < z$. Since $<$ is transitive, we get $x < z$, which by definition means $x \ll z$. So $\ll$ is transitive.

Thus, $(E, \ll)$ is an ordered set.

2. $(\mathbb{R}, <_f)$ where $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ and we say $x <_f y$ if $f(x) < f(y)$.

No! Note $-1 \neq 1$ but $f(-1) = f(1)$ since $(-1)^2 = 1^2 = 1$. Thus, none of $-1 <_f 1$, $-1 = 1$, or $1 <_f -1$ are true.

3. $(\mathbb{R}, <_g)$ where $g: \mathbb{R} \to \mathbb{R}$ with $g(x) = x^3$ and we say $x <_g y$ if $g(x) < g(y)$.

Yes! Note for $x, y \in \mathbb{R}$, exactly one of $x^3 < y^3$, $x^3 = y^3$, or $y^3 < x^3$ is true, since $g$ is injective. Thus exactly one of $x <_g y$, $x = y$, or $y <_g x$ is true.

Also, if $x <_g y$, $y <_g z$, then $x^3 < y^3$ and $y^3 < z^3$, so $x^3 < z^3$, and thus $x <_g z$.

4. $(S, <)$ where $S$ is the set of organisms in an ecosystem and $x<y$ if $y$ eats $x$.

No! Both parts of the definition of an order fail. For example, deer don't eat rabbits and rabbits don't eat deer, so none of the statements deer < rabbit, deer = rabbit, or rabbit < deer are true. In addition, it is not transitive, since for example, Barn owls eat mice, and mice eat seeds, but barn owls do not eat seeds.

5. $(S, <)$ where $S$ is the set of people in a family tree and $x<y$ if $x$ is a descendent of $y$.

No! Siblings are not descendants of each other so they are not comparable (Neither $x<y$ nor $y<x$).

6. $(S_1 \cup S_2, <)$ where $(S_1, <_1)$ and $(S_2, <_2)$ are disjoint ordered sets and $x<y$ if either $x, y \in S_1$ with $x <_1 y$; $x, y \in S_2$ with $x <_2 y$; or $x \in S_1$ and $y \in S_2$.

Yes! Let $x, y \in S_1 \cup S_2$

- If both $x, y \in S_1$, then exactly one of $x <_1 y$, $x = y$, or $y <_1 x$ is true, implying that exactly one of $x < y$, $x = y$, or $y < x$ is true.
- Similarly if both $x, y \in S_2$
- If $x \in S_1$ and $y \in S_2$, then by definition $x < y$ and $y \nless x$, and since $S_1 \cap S_2 = \emptyset$, $x \neq y$.
- Similarly if $y \in S_1$ and $x \in S_2$.

Thus, not matter what case, exactly one of $x<y$, $y<x$, or $x=y$ is true.

For transitivity, suppose $x<y$ and $y<z$. A similar checking of cases will show no matter where $x, y$, and $z$ live, $x<z$.

This order is called the <u>sum</u> of orders and the ordered set is sometimes written as $S_1 + S_2$.

7. $(\mathbb{R}^2, <)$ where $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ and $y_1 < y_2$.

No! Note that under this relation, none of $(1,3) < (4,2)$, $(1,3) = (4,2)$, or $(4,2) < (1,3)$ are true since $1 < 4$ but $2 < 3$.

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Determine if the following subsets of ordered sets are bounded above and/or below, and if so, if they have an infimum and/or supremum in that set.

8. $\{(-1)^n n \mid n \text{ positive integer}\}$ as a subset of $\mathbb{R}$.

This set is unbounded since for any $x \in \mathbb{R}$ you pick, if $x > 0$, then there exists $n \in \mathbb{N}$ with $n > x$. Then pick $N > n$ even, and so $(-1)^N N = N > n > x$. So this set is not bounded above.

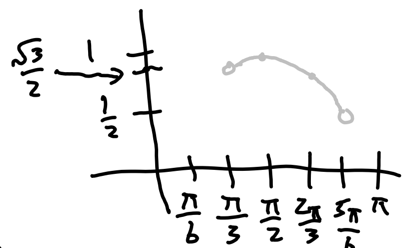
Similarly, you can show it is not bounded below.

9) $\operatorname{Im} f$ as a subset of $\mathbb{R}$, where
$f: (\frac{\pi}{3}, \frac{5\pi}{6}) \to \mathbb{R}$ defined by $f(x) = \sin x$.

$\operatorname{Im} f$ is bounded above and below, since for any real number $x$, $-1 \le \sin x \le 1$.

Note $\sup \operatorname{Im} f = 1$ since $f$ achieves its max at $\frac{\pi}{2}$ $\left(\sin\frac{\pi}{2} = 1\right)$.

From the graph of $\sin x$ on $\left(\frac{\pi}{3}, \frac{5\pi}{6}\right)$, we can see that $\frac{1}{2} < \sin x$ for every $x \in \left(\frac{\pi}{3}, \frac{5\pi}{6}\right)$.




Although we don't have the tools to prove it right now, since $\sin x$ is continuous and $\sin x \to \frac{1}{2}$ as $x \to \frac{5\pi}{6}$, this is the greatest lower bound of $\operatorname{Im} f$.

So $\inf \operatorname{Im} f = \frac{1}{2}$.

10) $\{2, 2.2, 2.22, 2.222, \dots\}$ as a subset of $\mathbb{R}$.

Let $S = \{2, 2.2, 2.22, 2.222, \dots\}$

Note $2 \le x \le 3$ for every $x \in S$, so this set is bounded above and below.

We have $\inf S = 2$ since $2$ is a lower bound of $S$, and for $a > 2$, $a$ can't be a lower bound of $S$ since $2 \in S$.

On the other hand, $\sup S = 2.\overline{2}$, since we always have $2.22\dots2 < 2.\overline{2}$, and

$\sup S > 2.\underbrace{22\dots2}_{k \text{ times}}$ for any $k$, since

$2.\underbrace{22\dots2}_{k+1 \text{ times}} \in S$ with $2.\underbrace{22\dots2}_{k+1 \text{ times}} > 2.\underbrace{22\dots2}_{k \text{ times}}$

11) $\{2, 2.2, 2.22, 2.222, \ldots\}$ as subset of $\mathbb{Q}$.

Again, let $S = \{2, 2.2, 2.22, 2.222, \ldots\}$.

Since $S$ is bounded in $\mathbb{R}$, it is also bounded in $\mathbb{Q}$.

Also, note $2$ and $2.\overline{2} = 2\frac{2}{9} = \frac{20}{9}$ are both rational, so $\inf S$ and $\sup S$ are in $\mathbb{Q}$.

# Ordered Sets and Bounds 2 solutions

1. We stated in the notes today that $\sqrt{2}$ is irrational. Prove it!

__Proof:__ Suppose by contradiction that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $p$ and $q$ are relatively prime (their greatest common divisor is $1$). Then

$$2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2 \Rightarrow p^2 \text{ is even}$$

$\Rightarrow p$ is even (since if $p$ were odd, then $p^2$ would be odd.)

So $p = 2m$ for some integer $m$.

$$\Rightarrow 2q^2 = (2m)^2 = 4m^2 \Rightarrow q^2 = 2m^2$$

$\Rightarrow q^2$ is even $\Rightarrow q$ is even.

So both $p$ and $q$ are even, which implies $2$ divides both $p$ and $q$, contradicting that $p$ and $q$ are relatively prime. Thus, we must conclude that $\sqrt{2}$ is irrational. $\boxtimes$

2. Does $\mathbb{R}^2$ with the dictionary order have the least upper bound property? Prove or provide a counterexample.

We claim $\mathbb{R}^2$ does not have the least upper bound property. Indeed, let

$$A = \{(x,y) \in \mathbb{R}^2 \mid x < 0\}$$

Then $A$ is bounded above since $(0,0)$ is an upper bound of $A$. However, $A$ does not have a least upper bound in $\mathbb{R}^2$. If it did, note that $\sup A$ could not be of the form $(0,y)$ for $y \in \mathbb{R}$, for even though $(0,y)$ is an upper bound of $A$, so is $(0,y-1)$, and $(0,y-1) < (0,y)$ since $0=0$ and $y-1<y$.

Thus, $\sup A$ would have to be of the form $(x,y)$ for $x<0$. But such a point in $\mathbb{R}^2$ would not be an upper bound of $A$, since then $(x,y+1) \in A$ and $(x,y+1) > (x,y)$. Thus, no least upper bound exists.

3. Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Define $-A = \{-x \mid x \in A\}$. Prove that $-A$ is bounded above and $\inf A = -\sup(-A)$.

__Proof:__ Since $A$ is bounded below, $\inf A$ exists in $\mathbb{R}$ and $\inf A \leq x$ for every $x \in A$

$\Rightarrow -\inf A \geq -x$ for every $x \in A$

$\Rightarrow -\inf A$ is an upper bound of $-A$.

So $-A$ is bounded above. We claim $-\inf A = \sup(-A)$. Indeed, if $y < -\inf A$, $\inf A < -y$. Then by definition of infimum, there exists $x \in A$ with $x < -y$. Then $-x \in -A$ with $-x > y$. So $y$ is not an upper bound of $-A$. So $-\inf A$ is the least upper bound of $-A$, i.e.

$$-\inf A = \sup(-A) \Rightarrow \inf A = -\sup(-A). \quad \boxed{\text{ℳ}}$$

4. Let $S$ be an ordered set that has the greatest lower bound property. Prove that $S$ has the least upper bound property.

Proof: The proof is similar to that of Theorem I. Let $E$ be a nonempty subset of $S$ that is bounded above. Let $U$ be the set of all upper bounds of $E$, which is nonempty since $E$ is bounded above. Then $U$ is bounded below since for every $u \in U$, $x \leq u$ for every $x \in E$ since $u$ is an upper bound of $E$. Since $S$ satisfies the greatest lower bound property, $\inf U$ exists in $S$. We claim $\sup E = \inf U$.

Suppose by contradiction $\inf U$ is not an upper bound of $E$. Then there exists $x \in E$ with $x > \inf U$. By definition of infimum, $x$ is not a lower bound of $U$. So there exists $u \in U$ with $u < x$. But since $u \in U$, $x \in E$, $u \geq x$, a contradiction! So $\inf U$ is an upper bound of $E$.

Second, suppose $x \in S$ with $x < \inf U$. Since $\inf U \leq u$ for all $u \in U$, $x \notin U$. So $x$ is not an upper bound of $E$.

So $\inf U$ is the least upper bound of $E$. So $\inf U = \sup E$.

Thus every nonempty bounded above subset of $S$ has a supremum in $S$, so $S$ has the least upper bound property. $\blacksquare$

5. Let $x$ be a positive real number and $n$ a positive integer.

(a) Let $A = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^n < x\}$. Show that $A$ has a supremum in $\mathbb{R}$.

Proof: Note $A$ is nonempty since we have:

$0 < \frac{x}{1+x} < 1 \Rightarrow \left(\frac{x}{1+x}\right)^n < \frac{x}{1+x}$. In addition,

$\frac{x}{1+x} < x \Rightarrow \left(\frac{x}{1+x}\right)^n < x \Rightarrow \frac{x}{1+x} \in A$.

In addition, $A$ is bounded above since $1+x$ is an upper bound of $A$:

If $t > 1+x$ then $t > 1 \Rightarrow t^n > t > 1+x > x$, i.e. $t^n > x$, so $t \notin A$ (This is the contrapositive of the definition of upper bound, which is equivalent)

Then by Theorem II $\sup A$ exists in $\mathbb{R}$. $\blacksquare$

(b) Prove that $b^n - a^n < (b-a)nb^{n-1}$ whenever $0 < a < b$.

<u>Proof</u>: We have $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$.

But $0 < a < b$ implies

$$b^{n-2}a < b^{n-2}b = b^{n-1}$$

$$b^{n-3}a^2 < b^{n-3}b^2 = b^{n-1}$$

$$\vdots$$

$$a^{n-1} < b^{n-1}$$

Thus $b^{n-1} + b^{n-2}a + \dots + a^{n-1} < \underbrace{b^{n-1} + b^{n-1} + \dots + b^{n-1}}_{n \text{ times}} = nb^{n-1}$

$\Rightarrow b^n - a^n < (b-a)nb^{n-1}$. ∎

(c) In the rest of this problem, let $y = \sup A$. Prove that $y^n \not< x$.

<u>Proof</u>: Assume by contradiction that $y^n < x$. Then $x - y^n > 0$. Since $n(y+1)^{n-1} > 0$, $\frac{x-y^n}{n(y+1)^{n-1}} > 0$.

Choose $h \in \mathbb{R}$ such that $0 < h < \min\left\{1, \frac{x-y^n}{n(y+1)^{n-1}}\right\}$

Then $0 < y < y+h$, so by part (b),

$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1}$ since $h<1$

$< \frac{x-y^n}{n(y+1)^{n-1}} n (y+1)^{n-1}$ since $h < \frac{x-y^n}{n(y+1)^{n-1}}$

$= x - y^n$

So $(y+h)^n - y^n < x - y^n \Rightarrow (y+h)^n < x$.

Since $y+h>0$, this implies $y+h \in A$. But then $y \geq y+h$ since $y = \sup A$, a contradiction since $h>0$. Thus $y^n \nless x$. $\boxed{\text{Ø}}$

(d) Prove that $y^n \ngtr x$ and conclude that there exists a positive $n^{th}$ root of $x$.

<u>Proof</u>: Assume by contradiction that $y^n > x$.

Let $k = \frac{y^n - x}{ny^{n-1}}$. Note $y^n < ny^n + x \Rightarrow y^n - x < ny^n$

$\Rightarrow \frac{y^n - x}{ny^{n-1}} < y \Rightarrow k < y \Rightarrow y - k > 0$.

We show $y-k$ is an upper bound of $A$ by showing if $t > y-k$, then $t \notin A$.

Indeed, if $t > y-k > 0$, then $t^n > (y-k)^n$

$\Rightarrow -t^n < -(y-k)^n$ and so

$$y^n - t^n < y^n - (y-k)^n < kny^{n-1} \quad \text{by part (b)}$$

$$= \frac{y^n - x}{ny^{n-1}} ny^{n-1} = y^n - x$$

Thus $y^n - t^n < y^n - x \Rightarrow t^n > x$, so $t \notin A$.

Thus $y-k$ is an upper bound of $A$.

But $k = \dfrac{y^n - x}{ny^{n-1}} > 0$ since $y^n > x$, so

$$y-k < y = \sup A, \text{ a contradiction.}$$

Hence $y^n \not> x$.

Since we also know by (c) that $y^n \not< x$, we must conclude $y^n = x$. In addition, $y$ is positive since $y \geq \frac{x}{1+x} \in A$ which is positive. Thus, $y$ is a positive $n^{th}$ root of $x$. $\blacksquare$

(e) Prove there is exactly one positive real $n^{th}$ root of $x$.

<u>Proof</u>: Assume by contradiction that $y_1$ and $y_2$ are positive $n^{th}$ roots of $x$ with $y_1 \neq y_2$. Then without loss of generality assume $y_1 < y_2$. Then $0 < y_1 < y_2$

which implies $y_1^n < y_2^n$. But $y_1^n = x = y_2^n$, so this implies $x < x$, a contradiction.

This shows at most one positive $n^{th}$ root of $x$ exists, and (d) showed at least one positive $n^{th}$ root of $x$ exists.

Therefore, $x$ has exactly one positive $n^{th}$ root. $\boxed{\text{//}}$