Ordered Sets (Rudin p. 3)

In math, we often like to put an order on things.

Examples of things we like to order:

- the real numbers $\mathbb{R}$ (e.g. $1.243 < 1.251$)
- words in the English language (alphabetical order)

An order, intuitively, is a way of saying some elements come before, or are "smaller" than, others.

Let's make this more precise:

Def 1: Let S be a set. An order on S is a relation (i.e, a specific way of comparing elements of S), which we denote by $<$, satisfying:

(i) If $x, y \in S$ with $x \neq y$, then exactly one of the following is true:

$$x < y \quad \text{or} \quad y < x$$

"one thing always comes before or after the other in the set"

(ii) If $x, y, z \in S$ with $x < y$ and $y < z$, then $x < z$ (transitivity)

"when ordering things, we don't want a rock-paper-scissors situation:

$\ldots < \text{rock} < \text{paper} < \text{scissors} < \text{rock} < \ldots$

no way to say which of rock, paper, scissors come first

Def 2: The pair $(S, <)$, where $<$ is an order on S, is called an ordered set.

An order on a set S gives us a way of saying which elements precede, or are smaller than, other elements.

Examples: 1 The real numbers $\mathbb{R}$ with $<$ defined by $x < y$ if $y - x$ is positive.

2 The rational numbers $\mathbb{Q}$, the integers $\mathbb{Z}$, and the natural numbers $\mathbb{N}$ with the same definition of $<$.

(3) Let $(S,<)$ be an ordered set.

Then the Cartesian product $S \times S$ can be ordered with the dictionary order or lexicographic order:

$$(x_1, y_1) < (x_2, y_2) \text{ if } x_1 < x_2 \text{ or if}$$
$$x_1 = x_2 \text{ and } y_1 < y_2$$

(Name comes from this is how we order words in a dictionary)

An order gives us a way of saying which elements in a set precede, or are smaller than, other elements of the set.

We can also think of an order as which elements are bigger:

Notation: For x, y in an ordered set S, we say $x > y$ if $y < x$.

We will use the notation $x \le y$ to mean $x < y$ or $x = y$. Similarly, $x \ge y$ will mean $x > y$ or $x = y$.

Given an ordered set S, we can look at the order restricted to a subset $E \subseteq S$

- For example, $\mathbb{Q} \subseteq \mathbb{R}$, or $(-\infty, 0] \subseteq \mathbb{R}$

Sometimes, subsets are bounded in the sense that they don't go past a certain point in the ordered set.

- For example, $(-\infty, 0] \subseteq \mathbb{R}$ doesn't go past 0.

<u>Def 3</u>: Given $E \subseteq S$, $(S, <)$ an ordered set, if there exists $y \in S$ such that $x \leq y$ for every $x \in E$, we say E is <u>bounded above</u>, and y is called an <u>upper bound</u> of E.

Similarly, E is <u>bounded below</u> if there exists $y \in S$ such that $x \geq y$ for every $x \in E$, and y is called a <u>lower bound</u> of E.

Examples: ① $\mathbb{Q} \subseteq \mathbb{R}$ is not bounded above or below

② $(-\infty, 0] \subseteq \mathbb{R}$ is bounded above by 0

③ Note so is $(-\infty, 0) \subseteq \mathbb{R}$ bounded above by 0 even though $0 \notin (-\infty, 0)$

④ $A = \{p \in \mathbb{Q} \mid 0 < p^2 < 2\} \subseteq \mathbb{Q}$ is bounded above by 1.5 $(1.5^2 = 2.25)$

Note 1.42, 1.415 are also upper bounds of A in $\mathbb{Q}$. $(1.42^2 = 2.0164,\ 1.415^2 = 2.002225)$

Is there a best upper bound we can find?

Def 4: Let E be a subset of an ordered set S with E bounded above. Suppose there exists $a \in S$ such that: ⓘ a is an upper bound of E

ⓘⓘ If $x \in S$ with $x < a$ then x is not an upper bound of E (no smaller number bounds E)

Then a is called the least upper bound of E or supremum of E and write

$$a = \sup E$$

Similarly, if E is bounded below, then if $z \in S$ satisfies (i) z is a lower bound of E

(ii) If $x \in S$ with $x > z$ then x is not a lower bound of E

then z is called a <u>greatest lower bound</u> or <u>infimum</u> of E, and write

$$z = \inf E.$$

<u>Examples:</u> ① $\sup(-\infty, 0] = 0$

② $\sup(-\infty, 0) = 0$

③ If we think of $A = \{p \in \mathbb{Q} \mid 0 < p^2 < 2\}$ as a subset of $\mathbb{R}$, then $\sup A = \sqrt{2}$

But $\sqrt{2}$ is irrational, so $\sup A$ does not exist in $\mathbb{Q}$.

Not every subset of an ordered set may have a supremum/infimum, even if that set is bounded above or below!

More on Ordered Sets/Bounds

- Recall we defined $A = \{p \in \mathbb{Q} \mid 0 < p^2 < 2\}$.
  This set had no least upper bound in $\mathbb{Q}$.
- Similarly, the set $B = \{p \in \mathbb{Q} \mid p > 0,\ p^2 > 2\}$ has no greatest lower bound in $\mathbb{Q}$.
- However, they do have a supremum/infimum, respectively, in $\mathbb{R}$ (it's $\sqrt{2}$).

It would be nice if we could guarantee that all bounded subsets of an ordered set S had a supremum/infimum in S.

__Def 5:__ An ordered set S is said to have the __least upper bound property__ if, for every nonempty subset $E \subseteq S$ that is bounded above, $\sup E$ exists in S.

Similarly, S is said to have the __greatest lower bound property__ if for every nonempty subset $E \subseteq S$ that is bounded below, $\inf E$ exists in S.

<u>Examples</u>: 1) As we saw with the examples of the sets A and B, $\mathbb{Q}$ does not satisfy the least upper bound/greatest lower bound properties.

2) However, we will see that $\mathbb{R}$ does satisfy the least upper bound/greatest lower bound properties.

There is a close relationship between supremum/infimums and the least upper bound/greatest lower bound properties. In fact,

<u>Thm I</u>: Every ordered set with the least upper bound property has the greatest lower bound property.

<u>Proof</u>: Let $(S, <)$ be an ordered set with the least upper bound property. We wish to show S has the greatest lower bound property.

So let $E \subseteq S$ be nonempty and bounded below. We wish to show $\inf E$ exists in S.

Our strategy will be to construct a nonempty bounded above set, which thus has a supremum in S, and whose supremum is the infimum of E.

Let L be the set of all lower bounds of E.

Since E is bounded below, E has a lower bound (i.e. an element $y \in S$ such that $y \leq x$ for every $x \in E$), so <u>L is nonempty</u>.

Second, given $x \in E$, we have, for any $\ell \in L$, $\ell \leq x$, since L is the set of lower bounds of E. So <u>L is bounded above</u>.

Thus, since S has the least upper bound property, $\sup L$ exists in S.

We claim $\sup L = \inf E$. This means we need to show $\sup L$ is a lower bound of E and if $x \in S$ with $x > \sup L$ then x is not a lower bound of E.

First, we show $\sup L$ is a lower bound of E. So suppose by contradiction there exists $x \in E$ with $x < \sup L$. By definition of supremum, x is not an upper bound of L. So there exists $l \in L$ with $x < l$. But l is a lower bound of E by definition of L. So $l \leq x$, a contradiction! Therefore, $\sup L$ must be a lower bound of E.

Second, suppose $x \in S$ with $x > \sup L$. Since $\sup L$ is an upper bound of L, $x \notin L$. So by definition of L, x is not a lower bound of E.

Hence, we have shown $\sup L = \inf E$.

Thus, every nonempty bounded below subset of S has an infimum in S. So S has the greatest lower bound property. $\blacksquare$

Ordered sets with the least upper bound property are important for proofs in real analysis.

In fact, the least upper bound property is essential to rigorously defining the real numbers $\mathbb{R}$.

What is $\mathbb{R}$? It's basically positive/negative numbers with an integer part and a (potentially) infinite decimal expansion. But even that's not rigorous enough since, for example, it does not account for the fact that $0.9999\ldots = 1$.

A rigorous definition of the real numbers is difficult and we won't go over it for now.

If you're curious, see p. 5-8 and 17-21 of Rudin for a rigorous treatment; for now, we will state the following theorem without proof:

Thm II: The real numbers have the least upper bound property.

Later on in the course, we will give an explicit construction of the real numbers.

Also, notice:

Cor: $\mathbb{R}$ satisfies the greatest lower bound property.

Proof: This follows from Thm I and the fact that $\mathbb{R}$ has the least upper bound property. ∎

Note that in contrast, as we showed earlier, $\mathbb{Q}$ does not have the least upper bound property. Thus, we like to use $\mathbb{R}$ rather than $\mathbb{Q}$ in analysis, even though $\mathbb{R}$ is a much more complicated set.

Let's see an example of how we can use the least upper bound property in $\mathbb{R}$.

Thm III: (The Archimedean Property)
If $x, y \in \mathbb{R}$ with $x > 0$, then there exists a positive integer n such that $nx > y$.

<u>Proof</u>: Let $A = \{nx \mid n \in \mathbb{N}\}$ $\{\mathbb{N} = \{m \in \mathbb{Z} \mid m > 0\}\}$

Suppose by contradiction that $nx \leq y$ for every positive integer n. Then y is an upper bound of A. Since $\mathbb{R}$ has the least upper bound property, A has a least upper bound $\sup A$ in $\mathbb{R}$.

Since $x > 0$, $-x < 0$ and $\sup A - x < \sup A$.

By definition of supremum, $\sup A - x$ is not an upper bound of A. Hence $\sup A - x < mx$ for some positive integer m. This implies

$$\sup A < mx + x = (m+1)x.$$

But $(m+1)x \in A$, contradicting that $\sup A$ is an upper bound of A.

Thus, there must be some positive integer n with $nx > y$.

$\square$